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A Green's Function Method for Treating Compton Frequency Redistribution

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A Green's Function Method for Treating Compton Frequency Redistribution

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Outline

- Review the Fokker-Planck equation for Compton scattering
- Properties of the exact Compton scattering kernel
- Green's function and moments for a Liouville differential equation
- Inference of Liouville coefficients to match Compton kernel moments
- Numerical comparison of Green's function with exact scattering kernel
- Advantages of Green's function method

Fokker-Planck = Kompaneets equation

$$\frac{1}{\chi_e c} \frac{\partial n}{\partial t} = \frac{kT}{mc^2} \frac{1}{x^2} \frac{\partial}{\partial x} \left[x^4 \left(\frac{\partial n}{\partial x} + n(n+1) \right) \right]$$

$$n(\nu) \equiv \frac{c^2 J_\nu}{2h\nu^3}$$

$$x \equiv \frac{h\nu}{kT}$$

Exact scattering kernel $R(\nu, \nu')$

$$\chi_T R(\nu, \nu') = N_e \frac{\nu}{\nu'} \frac{d\sigma}{d\nu}$$

$$\frac{1}{\chi_T c} \frac{\partial n_\nu}{\partial t} = \frac{c^2}{2h\nu^3} \int_0^\infty d\nu' \left\{ -R(\nu', \nu) \frac{\nu}{\nu'} J_\nu \left[1 + \frac{c^2 J_{\nu'}}{2h\nu'^3} \right] + R(\nu, \nu') J_{\nu'} \left[1 + \frac{c^2 J_\nu}{2h\nu^3} \right] \right\}$$

$$R(\nu, \nu') = \left(\frac{\nu}{\nu'} \right)^4 \exp \left[\frac{h(\nu' - \nu)}{kT} \right] R(\nu', \nu)$$

Moments of the scattering kernel

Symmetrized scattering kernel $r(\nu, \nu')$

$$R(\nu, \nu') = \frac{\nu^3 \exp(-h\nu/kT)}{\nu'} r(\nu, \nu'), \quad r(\nu', \nu) = r(\nu, \nu')$$

Evaluation of moments for $kT \ll mc^2, h\nu \ll mc^2$ —

$$A(\nu) = \int_0^\infty d\nu' \nu'^2 \exp\left(-\frac{h\nu'}{kT}\right) (\nu' - \nu) r(\nu', \nu) \approx \frac{kT}{mc^2} \left(4 - \frac{h\nu}{kT}\right) \nu$$
$$B(\nu) = \int_0^\infty d\nu' \nu'^2 \exp\left(-\frac{h\nu'}{kT}\right) (\nu' - \nu)^2 r(\nu', \nu) \approx \frac{2kT}{mc^2} \nu^2$$

Use of these moments in the Fokker-Planck method leads to Kompaneets' equation. Methods for obtaining accurate relativistic moments have been given by Prasad, *et al.*

Green's function of a Liouville differential equation

Liouville equation

$$L[y] = 0 \quad \text{with} \quad L[y] \equiv \frac{d}{dv} \left(p(v) \frac{dy}{dv} \right) + q(v)y$$

Green's function $G(v, v')$

$$L[y] = f(v) \quad \rightarrow \quad y(v) = \int_0^\infty dv' G(v, v') f(v')$$

$$\frac{\partial}{\partial v} \left(p(v) \frac{\partial G}{\partial v} \right) + q(v)G = \delta(v - v')$$

Moments of the Liouville Green's function

These integrals can be derived —

$$\int_0^\infty dv q(v) G(v, v') = 1$$

$$\int_0^\infty dv (v - v') q(v) G(v, v') = - \int_0^\infty dv \frac{dp}{dv} G(v, v')$$

$$\int_0^\infty dv (v - v')^2 q(v) G(v, v') = 2v' \int_0^\infty dv \frac{dp}{dv} G(v, v') - \int_0^\infty dv \frac{d}{dv} (2pv) G(v, v')$$

If $p(v) < 0$ and $|p(v)| \ll q(v)v^2$ then G is sharply peaked and

$$\begin{aligned} \int_0^\infty dv (v - v') q(v) G(v, v') &\approx -\frac{1}{q(v')} \frac{dp}{dv'} \\ \int_0^\infty dv (v - v')^2 q(v) G(v, v') &\approx -\frac{2p(v')}{q(v')} \end{aligned}$$

The Liouville operator that matches Compton scattering moments for $kT \ll mc^2$

Comparing the Green's function moments to the small- T Compton moments suggests

$$\begin{aligned} q(\nu) &= \nu^2 \exp\left(-\frac{h\nu}{kT}\right) \\ p(\nu) &= -\frac{kT}{mc^2} \nu^4 \exp\left(-\frac{h\nu}{kT}\right) \end{aligned}$$

and these satisfy the conditions for G to be sharply peaked provided $kT \ll mc^2$. So

$$\begin{aligned} L[G(\nu, \nu')] &\equiv -\frac{kT}{mc^2} \frac{\partial}{\partial \nu} \left[\nu^4 \exp\left(-\frac{h\nu}{kT}\right) \frac{\partial G(\nu, \nu')}{\partial \nu} \right] + \\ &\quad \nu^2 \exp\left(-\frac{h\nu}{kT}\right) G(\nu, \nu') = \delta(\nu - \nu') \end{aligned}$$

Expression for G in terms of special functions

The differential equation obtained for G is a variation of the confluent hypergeometric equation. With the notation

$$x \equiv \frac{h\nu}{kT}, \quad \theta \equiv \frac{kT}{mc^2}, \quad s = -\frac{3}{2} + \sqrt{\frac{9}{4} + \frac{1}{\theta}},$$

G is given by

$$G(\nu, \nu') = \frac{\Gamma(s)}{\Gamma(2s+4)} \frac{mc^2}{kT} \left(\frac{h}{kT} \right)^3 (xx')^s M(s, 2s+4, x_{<}) U(s, 2s+4, x_{>})$$

M and U are Kummer (confluent hypergeometric) functions

Numerical solutions for the Green's function

The Liouville equation for G can be put into a conservative finite difference form with the method of Larsen, Levermore, Pomraning and Sanderson (1985). It reduces to a problem of finding the inverse of a tri-diagonal matrix; the matrix is almost identical to the one obtained by Larsen, *et al.*

$$H(\nu, \nu') = \nu^2 \exp\left(-\frac{h\nu}{kT}\right) G(\nu, \nu') \approx \frac{\nu'}{\nu} R(\nu, \nu') = \frac{N_e}{\chi_T} \frac{d\sigma}{d\nu}$$

The tri-diagonal matrix U that yields H by inversion

$$H = \frac{h}{kT} U^{-1}, \quad U = (a_j \quad b_j \quad c_j)$$

$$a_j = -\frac{\theta e^{x_{j-1}}}{x_{j-1}^2} \frac{x_{j-1/2}^4}{e^{x_j} - e^{x_{j-1}}}$$

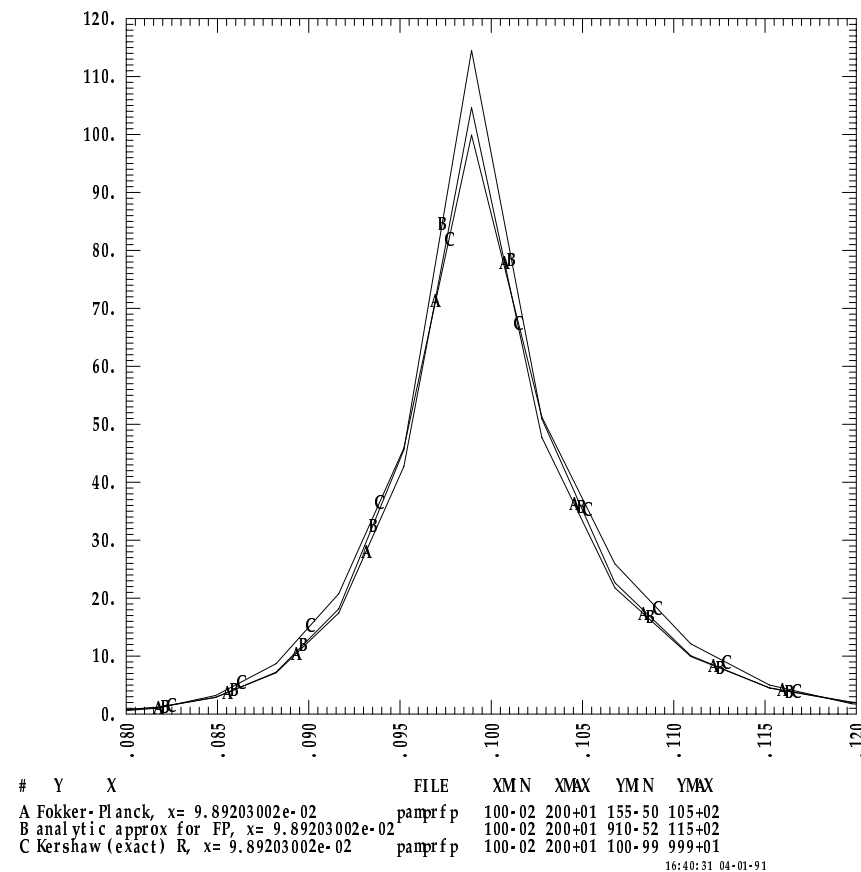
$$b_j = \frac{\theta e^{x_j}}{x_j^2} \left[\frac{x_{j-1/2}^4}{e^{x_j} - e^{x_{j-1}}} + \frac{x_{j+1/2}^4}{e^{x_{j+1}} - e^{x_j}} \right] + x_{j+1/2} - x_{j-1/2}$$

and

$$c_j = -\frac{\theta e^{x_{j+1}}}{x_{j+1}^2} \frac{x_{j+1/2}^4}{e^{x_{j+1}} - e^{x_j}}$$

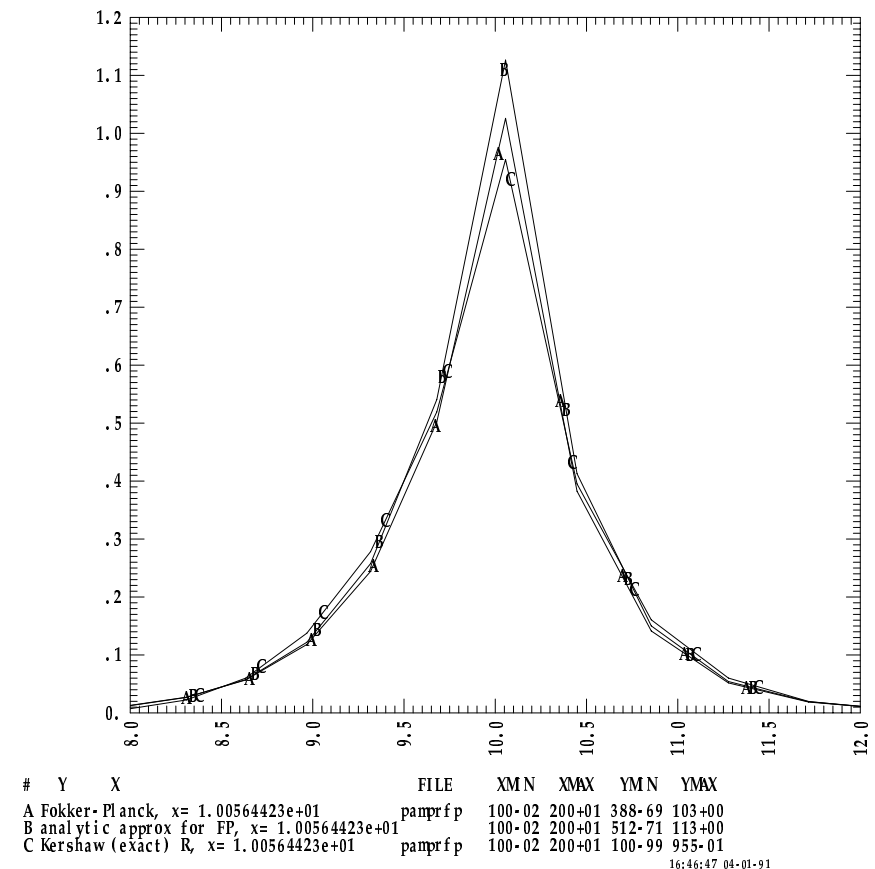
Sample calculations of R

Here for $h\nu' \approx 0.1 \text{ keV}$, $kT = 1 \text{ keV}$. A is the Green's function method, B is an analytic approximation, and C is exact.



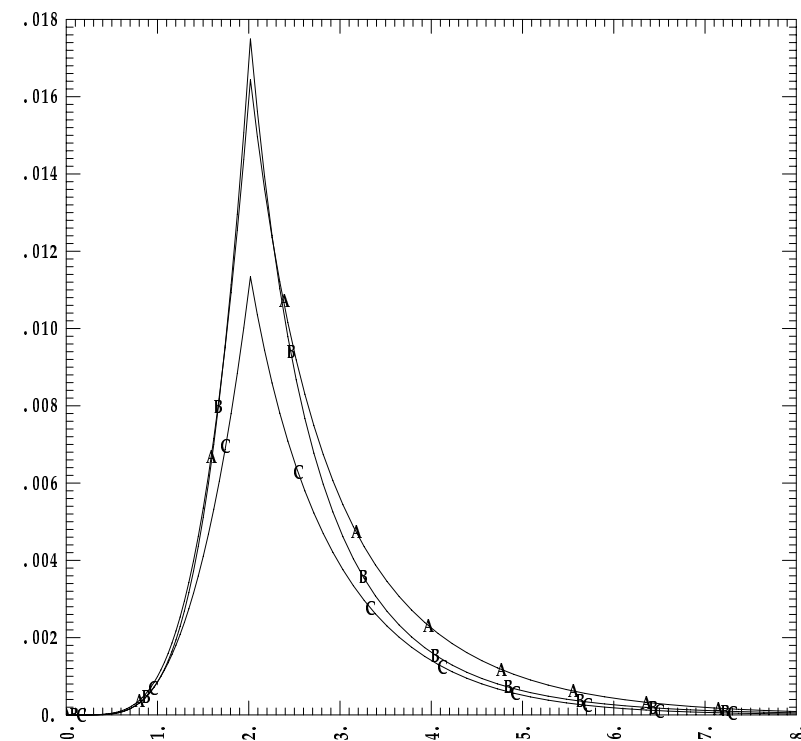
More sample calculations of R — at larger ν'

For $h\nu' \approx 10\text{keV}$, $kT = 1\text{keV}$. A is the Green's function method, B is an analytic approximation, and C is exact.



More sample calculations of R – at large T and ν'

For $h\nu' \approx 100 \text{ keV}$, $kT = 50 \text{ keV}$.
 A is the Green's function method,
 B is an analytic approximation,
 and C is exact.



#	Y	X	FILE	XMN	XMAX	YMN	YMAX
A	Fokker-Planck, $x = 2.01858044e+00$		pamprfp	200-03	200+01	127-17	164-02
B	analytic approx for FP, $x = 2.01858044e+00$		pamprfp	200-03	200+01	369-17	175-02
C	Kershaw (exact) R , $x = 2.01858044e+00$		pamprfp	200-03	200+01	369-76	113-02

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Advantages of the Green's function method

- The approximate value of the scattering kernel can be found by inverting a tri-diagonal matrix – a few FLOPS per matrix element
- The action of the kernel on a radiation spectrum can be found by solving a tri-diagonal system, and the kernel need not be found
- The solution of an implicit equation for the spectrum is another tri-diagonal system
- The last two operations are much faster than any full matrix calculations — the same as Fokker-Planck solutions
- By using the Prasad, *et al.*, diffusion coefficient the results can be made more accurate for large T